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LETTER TO THE EDITOR

Partition function of a finite Ising model on a torus

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Abstract. The detailed proof presented in an earlier paper, justifying Vdovichenko's method which gives an exact expression of the free energy of the Ising model in a plane, is extended to the system on a torus. This justifies the exact expression of the free energy in the form given by Vdovichenko for the Ising model on the general two-dimensional lattice in the thermodynamic limit.

A number of authors (Vaks *et al* 1966, Bryksin *et al* 1980, Kitatani *et al* 1985, Morita 1986b) applied the exact expression of the free energy of the Ising model in the form given by Vdovichenko (1965), for various systems on two-dimensional lattices. In examining Vdovichenko's derivation of the free energy, Morita (1986a) found difficulty in obtaining the final result for the general two-dimensional lattice because the exact partition function was obtained for a system in a plane but not on a torus. That paper will be referred to as I in the following. For the finite system on a square lattice on a torus, an exact expression of the partition function was given by Kaufman (1949). Potts and Ward (1955) were able to express it in terms of four determinants of large matrices. The purpose of the present letter is to extend the derivation given in I to give the exact partition function of the Ising model on the general two-dimensional lattice on a torus.

We start with the following expression of the partition function for the finite Ising model on a two-dimensional lattice of N lattice sites:

$$Z = 2^N \prod_{(j,k)} \cosh(\beta J_{jk}) Z_1(1) \tag{1}$$

where

$$Z_1(t) = [1 + \{\text{the sum of all those single-bonded diagrams on the lattice, that each lattice site is connected to none or an even number of bonds connecting nearest-neighbour sites}\}]. \tag{2}$$

Here the product in the second factor on the right-hand side of (1) is over all the pairs of nearest-neighbour lattice sites on the lattice and $\beta = 1/k_B T$, k_B is the Boltzmann constant and T is the temperature. A diagram here represents the product of factors for the bonds; the factor is $\tanh(\beta J_{jk})t$ for the bond connecting jth and kth lattice sites, where J_{jk} is the exchange integral between these lattice sites. Here t is a parameter introduced for the convenience of the proof.

We can rewrite $Z_1(t)$ as a sum of products of non-crossing loops as in I. The diagrams are now on a lattice on the torus. A loop may wind the torus. In order to distinguish two directions of winding, we call them the x and y direction, respectively. We call a loop with a net rotation in either direction a winding loop. We shall call a loop winding odd times in the x direction and no or even times in the y direction an o–e winding loop, a loop winding odd times in the y direction and no or even times in the x direction an e–o winding loop and a loop winding odd times both in the x and y direction an o-o winding loop. Otherwise, a winding loop is called an e–e winding loop, which always involves a crossing and does not appear in $Z_1(t)$.

In the present letter, we assume that a diagram represents the product of loops, divided by the symmetry number of the diagram as a whole, and a loop represents the product of the following four factors:

(i) -1, if the total number of crossings within the loop is odd;

(ii) the product $\Pi_{(j,k)}[\tanh(\beta J_{jk})t]^{m_{jk}}$, where m_{jk} is the total number of times that the bond between the jth site and the kth site is passed along the loop;

(iii) the inverse of the symmetry number of the loop;

(iv) -1, if the loop is an o-e, e-o or o-o winding loop.

$Z_1(t)$ is now expressed as

$$Z_1(t)=\Sigma_{00}-\Sigma_{10}-\Sigma_{01}-\Sigma_{11} \tag{3}$$

where Σ_{10}, Σ_{01} and Σ_{11} are the sums of the diagrams in which there are an odd number of o–e winding loops, an odd number of e–o winding loops and an odd number of o–o winding loops, respectively, and Σ_{00} is the sum of the remaining diagrams in $Z_1(t)$. The minus signs in front of the last three terms in (3) are due to (iv). We now consider the following quantity:

$$Z_{00}(t)=\Sigma_{00}+\Sigma_{10}+\Sigma_{01}+\Sigma_{11}. \tag{4}$$

We shall see that if the quantity which is expressed by $\det(\boldsymbol{I}-\boldsymbol{\Lambda})$ in I is calculated for a system on a torus, it is equal to $Z_{00}(1)^2$, but not to $Z_1(1)^2$.

We first show that $Z_{00}(t)$ is equal to $Z_2(t)$ which is defined by

$$Z_2(t)=\exp\{\text{the sum of all the loops on the torus}\}. \tag{5}$$

We follow the arguments in § 2 of I. We expand the exponential in (5). All the diagrams involved in $Z_{00}(t)$ given by (4) appear in the expansion. The arguments to show the cancellation of all the other diagrams in the expansion proceed as in I.

Two diagrams to be cancelled with each other are different only in the way of connection at one site and have numbers of crossings which are different by one. If the total numbers of loops in the two diagrams are equal with each other, we only have a change of factor -1 due to (i), resulting in the cancellation. If the change of the connection reduces a loop in one of the diagrams into two loops in the other, firstly we have -1 due to (i) as in the above case, and secondly the change of the sign factors due to (iv) as given in table 1. We observe that the total number of crossings between the two loops in the latter diagram is odd if these two loops are o–e and e–o, or o–e and o–o, or e–o and o–o winding loops, and it is even otherwise; see the appendix for an argument to show this fact. If this number is odd, we have an additional factor -1, since the factor -1 is associated with these crossings within a loop before the change, because of (i), but no sign factor is associated with the crossings between two separate loops after the change. Thus we conclude the cancellation for all the cases, obtaining

$$Z_{00}(t)=Z_2(t). \tag{6}$$

Table 1. Comparison of the sign factors due to (iv), for two diagrams of a loop and of two loops, to be cancelled with each other. Here o–o, o–e and e–o denote an o–o, o–e and e–o winding loop, respectively, and e–e represents an e–e winding as well as a non-winding loop.

Type of a loop	Types of two loops		Difference in the factors due to (iv)
o–o	o–o	e–e	1
o–o	o–e	e–o	-1
o–e	o–o	e–o	-1
o–e	o–e	e–e	1
e–e	o–o	o–o	$(-1)^2$
e–e	o–e	o–e	$(-1)^2$
e–e	e–e	e–e	1

We next consider the directed loops, rewriting (5) as follows:

$$Z_2(t)^2 = \exp\{\text{the sum of all the directed loops on the torus}\}. \qquad (7)$$

We compare the exponent of (7) with $(1/n)\,\mathrm{Tr}\,\Lambda^n t^n$, where Λ is the matrix which induces the random walks on the torus. We use i and i' to label a step of a random walk, discriminating between the nearest-neighbour pair of sites as well as the direction. The (i, i') element of Λ is

$$a_{i,i'} = x_i p_i \theta_{i,i'} \qquad (8)$$

if step i can follow i' and $|\phi_{i,i'}| < \pi$ and $a_{i,i'} = 0$ otherwise, where $\phi_{i,i'}$ is the angle of direction of step i relative to the direction of i' and $|\phi_{i,i'}| \leq \pi$ and $\theta_{i,i'} = \exp(\mathrm{i}\phi_{i,i'}/2)$. $x_i = \tanh(\beta J_{jk})$ if the step i is between the lattice sites j and k. In this case, $p_i = 1$ for all i.

All the loops on the torus appear in both members to be compared. We first consider the case that the sum of $\phi_{i,i'}$ is zero. We then have a winding loop. If it is o–e, e–o or o–o, we have an even number of crossings and if it is e–e, an odd number of crossings. (i) and (iv) give a factor -1 for all these cases. This factor -1 does not appear in Λ^n. In the case of a non-winding loop without crossing, we have a factor 1 in (i) and (iv), but -1 in Λ^n due to the product of $\theta_{i,i'}$. In every addition of a crossing, we have a factor -1 due to (i), which is taken into account in the product of $\theta_{i,i'}$ in Λ^n. Thus they have different signs for all the loops. Hence we have

$$\begin{aligned} Z_2(t)^2 &= \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\,\Lambda^n t^n\right) \\ &= \det(\boldsymbol{I} - \Lambda t). \end{aligned} \qquad (9)$$

See I for the roles played by the parameter t in obtaining (9).

The remaining argument is the same as in Potts and Ward (1955). We change the sign of the interactions in a row or in a column or in both, defining $Z_{10}(t)$, $Z_{01}(t)$ and $Z_{11}(t)$ by

$$\begin{aligned} Z_{10}(t) &= \Sigma_{00} - \Sigma_{10} + \Sigma_{01} - \Sigma_{11} \\ Z_{01}(t) &= \Sigma_{00} + \Sigma_{10} - \Sigma_{01} - \Sigma_{11} \\ Z_{11}(t) &= \Sigma_{00} - \Sigma_{10} - \Sigma_{01} + \Sigma_{11}. \end{aligned} \qquad (10)$$

Now comparing (3) with (4) and (10), we express $Z_1(t)$ as follows:

$$Z_1(t)=\tfrac{1}{2}[-Z_{00}(t)+Z_{10}(t)+Z_{01}(t)+Z_{11}(t)]. \tag{11}$$

We define $\Lambda_{\sigma\sigma'}$ for $\sigma=0$, 1 and $\sigma'=0$, 1 by choosing $p_i=(-1)^{\sigma}$ if i is a step on the column for which the sign of interaction is changed, $p_i=(-1)^{\sigma'}$ if i is a step on the row for which the sign of interaction is changed and $p_i=1$ for all other steps i. Then $\Lambda_{00}=\Lambda$ and

$$Z_{\sigma\sigma'}(t)^2=\det(I-\Lambda_{\sigma\sigma'}t). \tag{12}$$

For a finite Ising model on the general two-dimensional lattice on the torus, we obtain the partition function expressed by (1) with (11) and (12).

We shall now consider a translationally symmetric system of $N'=K\times L$ unit cells on a torus. A unit cell is labelled by (k, l) where $k=1,2,\ldots,K$ and $l=1,2,\ldots,L$, and unit cells $(k+1, l)$ and $(k, l+1)$ are derived from the unit cell (k, l) in the x and y direction, respectively, where $K+1$ and $L+1$ are identified with 1. When there are M pairs of nearest-neighbour lattice sites on the lattice, there are $2M/N'$ kinds of steps of random walks, for a unit cell. We shall label them by μ and ν, where $\mu, \nu=1,2,\ldots,2M/N'$.

We note here that there exist various choices of $p_i\theta_{i,i'}$, provided that the product of $p_i\theta_{i,i'}$ along any loop is not changed (Bryksin *et al* 1980). For instance, for the calculation of $\det(I-\Lambda_{\sigma\sigma'})$, we can choose p_i for every step i, not only for a step on a row or a column, to be

$$p_i=\exp[\mathrm{i}\pi(\sigma\xi_\mu/K+\sigma'\eta_\mu/L)] \tag{13}$$

where ξ_μ and η_μ denote $k''-k$ and $l''-l$, respectively, when step i is labelled by (k, l, μ) and is one from a site in the unit cell (k, l) to a site in a unit cell (k'', l''). Then $\Lambda_{\sigma\sigma'}$ have the translational and rotational symmetry in the x and y directions and we can apply the arguments in § 4 in I to express $\det(I-\Lambda_{\sigma\sigma'})$ in terms of small determinants for a unit cell. In place of (22) in I, we have

$$\det(I-\Lambda_{\sigma\sigma'})=\prod_{\kappa=1}^{K}\prod_{\kappa'=1}^{L}\det(\tilde{t}_{\mu\nu}(\pi(2\kappa+\sigma)/K, \pi(2\kappa'+\sigma')/L)) \tag{14}$$

where

$$\tilde{t}_{\mu\nu}(\psi, \psi')=\delta_{\mu\nu}-\tanh(\beta J_\mu)\theta'_{\mu\nu}\exp(\mathrm{i}\xi_\mu\psi+\mathrm{i}\eta_\mu\psi'). \tag{15}$$

Here J_μ denotes the interaction of the two sites on both sides of a step labelled by (k, l, μ). $\theta'_{\mu\nu}$ is equal to $\exp(\mathrm{i}\phi_{\mu\nu}/2)$ if there exists a step which is labelled by (k', l', ν) and can just precede the step (k, l, μ) and $|\phi_{\mu\nu}|<\pi$, where $\phi_{\mu\nu}$ is the angle of the step (k, l, μ) relative to the step (k', l', ν), $\theta'_{\mu\nu}=0$ if otherwise. For a system on the square lattice, the expression (1) with (11), (12) and (14) is equivalent to the result of Kaufman (1949), and to the result of Potts and Ward (1955) above the critical temperature.

Substituting (14) into (12), we observe that $\ln|Z_{\sigma\sigma'}(1)|$ is estimated as

$$\frac{1}{N'}\ln|Z_{\sigma\sigma'}(1)|=\frac{1}{N'}\ln|Z_{00}(1)|+\mathrm{O}(1/K)+\mathrm{O}(1/L). \tag{16}$$

We denote the ratio of $Z_1(1)$ and $Z_{00}(1)$ by C, i.e.

$$Z_1(1)=Z_{00}(1)C. \tag{17}$$

Since (1) shows that $Z_1(1)$ is positive, C cannot be zero. By using (16) and (17) in

(11), we obtain

$$\frac{1}{N'}\ln|C| = \mathrm{O}(1/K) + \mathrm{O}(1/L). \tag{18}$$

As a result, we have

$$\frac{1}{N'}\ln Z_1(1) = \frac{1}{N'}\ln|Z_{00}(1)| + \mathrm{O}(1/K) + \mathrm{O}(1/L). \tag{19}$$

Substituting this into (1) and taking the limit as $K\to\infty$ and $L\to\infty$, we obtain

$$\begin{aligned} -\beta f &= \lim_{\substack{K\to\infty\\ L\to\infty}} \frac{1}{N}\ln Z \\ &= \ln 2 + \frac{1}{2M'}\sum_{\mu}\ln J_\mu + \frac{1}{8\pi^2 M'}\int_0^{2\pi} \mathrm{d}\psi \int_0^{2\pi} \mathrm{d}\psi' \ln\det(\tilde{t}_{\mu\nu}(\psi,\psi')) \end{aligned} \tag{20}$$

where $M' = N/N'$ is the total number of lattice sites per unit cell.

(20) is the result which we obtain when we simply apply the result of Vdovichenko's method to a system on a torus and take the thermodynamic limit. We thus justify the use of of Vdovichenko's result in the thermodynamic limit for the Ising model on the general two-dimensional lattice.

Appendix. Parity of the number of crossings between two loops

We consider two loops L and L' on a torus. We denote the total numbers of windings in the x and y direction of the loops by m and n for L and by m' and n' for L'. We represent the torus by a rectangle of sides X and Y in the x and y direction, respectively, in the plane. We draw the loop L on the rectangle and its repetitions as shown in figure 1. We consider a point with coordinates (x, y) which is on L', but not on L, and

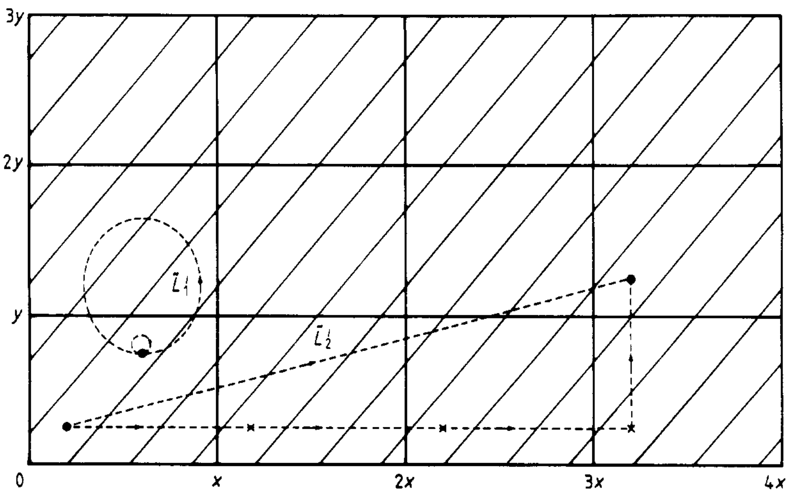

Figure 1. Representation of loops L and L' in the plane. L for $m = 2$ and $n = 3$ is shown by thin solid lines. $\tilde{L}_1'$ represents an L' with $m' = n' = 0$, which is deformed to a tiny circle. $\tilde{L}_2'$ represents another L' with $m' = 3$ and $n' = 1$, which is deformed to the line via three crosses.

represent the loop L' by a trajectory $\tilde{L}'$ from the point (x, y) to the point $(x+m'X, y+n'Y)$ in the plane, as shown in figure 1. Each of the crossings between L and L' on the torus appears once and only once as a crossing of the trajectory $\tilde{L}'$ with the lines for L in figure 1.

We now note that the parity of the total number of crossings between two loops on a torus does not change if we change the forms of the loops continuously. In fact, we observe that if we change the form of $\tilde{L}'$ in the plane, without changing the terminal positions (x, y) and $(x+m'X, y+n'Y)$, the change of the total number of crossings with the lines for L is even. If $m'=n'=0$, then L' is non-winding and we can deform $\tilde{L}'$ to a tiny circle near (x, y), reducing the total number of the crossings to zero, and hence the parity must be zero. If L' is a winding loop, we deform $\tilde{L}'$ to the trajectory described by the coordinates $r(t)$ for $0 \leqslant t \leqslant m'+n'$, where

$$r(t)=\begin{cases}(x+tX, y) & 0 \leqslant t \leqslant m' \\ (x+m'X, y+(t-m')Y) & m' \leqslant t \leqslant m'+n'.\end{cases}$$

At $0<t<m'$, the trajectory traverses a distance $m'X$ in the x direction, when the number of crossings in every distance X is equal to n or different from n by an even number. Hence the parity of the number of the crossings in this part of the trajectory is equal to the parity of the product $m'n$. At $m'<t<m'+n'$, the trajectory traverses a distance $n'Y$ in the y direction and the parity of the total number of the crossings in this part is equal to that of $n'm$. By summing these, we obtain the result that the parity of the number of the crossings between the two loops L and L' on a torus is equal to the parity of the sum $m'n+n'm$. Thus the parity is odd if the two loops are o–e and e–o, o–e and o–o, or e–o and o–o winding loops. It is even if otherwise.

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